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On the Singularities of Curves of Double Curvature.

By Henry B. Fine.

The elements of a curve of double curvature may be either the points of which it is the carrier or the lines or planes by which it is enveloped. We shall find it convenient to call any curve C the *point-curve*, *line-curve*, or *plane-curve* C, according as it is regarded under the first, second, or third of these aspects.

Each of these curves—if analytical—can be represented in the neighborhood of any element by an expansion in positive integral powers of a parameter with Grassmann units for coefficients (§ 1); these units being points in the series which represents the point-curve, in part lines, in part linear complexes in the series for the line-curve, and planes in the plane-curve series.

A curve being thus represented, every projective singularity possible to a single curve element finds expression in linear relations among the coefficients of the opening terms in its expansion.

Three classes of such relations among the coefficients of the point-curve or plane-curve expansions are supposable, and these I make the definitions of a 1st, 2d and 3d class point singularity, and of a 1st, 2d and 3d class plane singularity (§§ 2, 3), viz: I define a point as singular in the 1st class to the $\varkappa_1$ order when all the coefficients of its expansion through that of the $\varkappa_1$ power of the parameter are congruent; as singular in the 2d class to the $\varkappa_2$ order when the coefficients, through that of the $\varkappa_1 + \varkappa_2 + 1$ power of the parameter are linearly derivable from the leading coefficient and that of the $\varkappa_1 + 1$ power; as singular in the 3d class to the $\varkappa_3$ order when the coefficients, through that of the $\varkappa_1 + \varkappa_2 + \varkappa_3 + 2$ power of the parameter, are linearly derivable from the coefficients of the 0, $\varkappa_1 + 1$, and $\varkappa_1 + \varkappa_2 + 2$ powers; and have reciprocal definitions for a 1st, 2d and 3d class plane singularity with order indices μ_1, μ_2, μ_3.

The indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ play precisely the same rôle in the theory of higher singularities of curves of double curvature as do the "cuspidal" and "inflexional" indices (Cayley) in the corresponding theory for curves in a plane.*

Thus, for a pair of associated point and plane elements of any curve C, $\varkappa_1$ is equal to μ_3, $\varkappa_2$ to μ_2 and $\varkappa_3$ to μ_1, (§ 4).

Of a set of associated point, line and plane elements of the curve—P, l and ε—$\varkappa_1$ connects itself most closely with P, $\varkappa_2$ with l and $\varkappa_3$ with ε; these numbers being indeed the degrees to which P, l and ε respectively are stationary (§§ 2, 3, 6).

At P, a point tracing out the point-curve maintains the direction of its motion in l if $\varkappa_1$ be even, but reverses it if $\varkappa_1$ be odd; and the generating line and plane are controlled in like manner by $\varkappa_2$ and $\varkappa_3$. The eight combinations of odd and even values of the three numbers $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ give rise to the eight characteristic shapes—first noticed by von Standt†—possible to a curve of double curvature at any element (§§ 5, 6).

Finally $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are respectively the number of simple stationary points, lines and planes to which, in its effect on the order, class, rank and genus of C, the singularity is equivalent‡ (§ 6).

As there are six linearly independent line elements in space, the number of singularity-defining relations supposable among the coefficients of the line-curve series is five. It is found (§ 6) that these are all controlled by the indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ when $\varkappa_1$ and $\varkappa_3$ are unequal, the first four of them when $\varkappa_1 = \varkappa_3$. An *independent* line singularity is possible, therefore, when $\varkappa_1 = \varkappa_3$ and only then. I call it the line class singularity and assign it the order index λ. This singularity is, geometrically speaking, that more consecutive elements of a line-curve, or developable, belong to the same linear complex than are forced into it by $\varkappa_1$, $\varkappa_2$, $\varkappa_3$.

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* Plücker was the first to define the higher singularities of algebraic plane curves by indices. He used two, an order and a class index. Prof. Cayley's cuspidal and inflexional indices are Plücker's order and class indices diminished each by unity. They give the number of simple cusps and inflexions which are to be regarded as entering into the composition of the higher singularity. Vid. Plücker's Theorie der algebraischen Curven, p. 205, and Cayley in Quarterly Journal, Vol. 7; also Smith in Proceedings of the London Math. Society, Vol. 6.

† Geom. d. Lage. Rückkehrelemente.

‡ There enter also into the composition of higher singularities in general, double points, double planes, etc., due to the mutual intersections of the projections of the several "partial branches" which unite in the singular element (as do the leaves of a Riemann surface at a spiral point). These we leave out of sight; the series with which we operate represent but single partial branches, and regarded as an element of such a branch, the singularity is completely defined by the indices κ_1, κ_2, κ_3, λ.

The highest projective singularity possible to an element of C is defined by its indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, λ.

In § 7 I have expressed the conditions of the singularity $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ in terms of the coefficients in the expansions $\phi_i(t)$, $\psi_i(t)$ by which the point- or plane-curve is defined; and in § 8 the same conditions in terms of position relations among corresponding elements of the curve $\phi_i(t)$ (or $\psi_i(t)$) and its "differential curves" $\phi_i'(t)$, $\phi_i''(t)$, etc., ... ($\psi_i'(t)$, $\psi_i''(t)$, etc., ...). The use of the differential curves for this purpose was suggested to me by my friend Dr. Study of Leipsic, to whom I am indebted for other valuable suggestions, in particular for calling my attention to the singularity of six consecutive tangents lying in the same linear complex—the simplest case of the line-class singularity.

As Grassmann is not very generally read, I have presupposed an acquaintance with but the fundamental ideas of his algebra.* For his terms Punkt, Linientheil, Liniensumme, Ebenentheil, I have used simply point, line, linear complex, plane; for, though somewhat inexact, this terminology is simple and not liable to cause confusion.

The entire investigation admits of immediate extension to curves in space of n dimensions. It is only necessary to enlarge sufficiently the order of our region (Stufe). There are n classes of point and of "plane" singularities in a space of n dimensions; the one class for a space of one dimension being the Riemann spiral point (Verzweigungspunkt).

§ 1.

The equations

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t),$$

where x_i are homogeneous point coordinates and ϕ_i analytical functions of the form $a_{i0} + a_{i1}(t - t_0) + a_{i2}(t - t_0)^2 + a_{i3}(t - t_0)^3 + \text{etc.} \dots$,
which converge within a circle of the complex plane of t whose centre is t_0, define within this circle an analytical point-curve of double curvature. It is assumed that, generally speaking, to each point (x) but one value of t belongs. Of special interest among the analytical curves are the algebraic. These are defined in the first instance by equations of the form

$$x_1 : x_2 : x_3 : x_4 = \phi_1(\xi, \eta) : \phi_2(\xi, \eta) : \phi_3(\xi, \eta) : \phi_4(\xi, \eta),$$

the ϕ_i being integral rational functions of ξ and η, and ξ and η being connected by an algebraic relation $f(\xi, \eta) = 0$. But this algebraic relation enables us by a simple substitution to reduce the equations given to the general form $\rho x_i = \phi_i(t)$. For at a multiple point of the μ^{th} order on the curve $f(\xi, \eta) = 0$—we suppose its coordinates to be ξ_0, η_0—each of the μ values which $\eta - \eta_0$ can take may be developed in positive integral powers of $(\xi - \xi_0)^{\frac{1}{\mu}}$ if not in such powers of a lower root of $\xi - \xi_0$; so that it is only necessary to substitute $t - t_0$ for this radical to accomplish the desired reduction.*

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* For a brief exposition of these, see an article by Prof. Hyde in Vol. VI of this Journal. Or see Grassmann: Ausdehnungslehre of 1844; Ausdehnungslehre of 1862, Part 1, Chapter 5.

If E_1, E_2, E_3, E_4 be Grassmann points, each of the weight 1, and situate one at each of the angular points of the tetraedron $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, the point (x) is $P = x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4$. For any point in space can be derived linearly from E_1, E_2, E_3, E_4, so that $P = \alpha_1E_1 + \alpha_2E_2 + \alpha_3E_3 + \alpha_4E_4$. Taking the progressive product of both sides of this equation into $E_2E_3E_4$, $PE_2E_3E_4 = \alpha_1E_1E_2E_3E_4$. Each of the products $PE_2E_3E_4$ and $E_1E_2E_3E_4$ is scalar, and their ratio is that of the tetraedra $PE_2E_3E_4$ and $E_1E_2E_3E_4$; α_1, therefore, is in constant ratio to the perpendicular distance of P from the tetraedron face $E_2E_3E_4$ and is consequently equal to x_1. For the same reason $\alpha_2 = x_2$, $\alpha_3 = x_3$, $\alpha_4 = x_4$.

It follows, therefore, that within the circle of convergence common to the four functions $\phi_i(t)$,

$$P = \phi_1(t)E_1 + \phi_2(t)E_2 + \phi_3(t)E_3 + \phi_4(t)E_4$$

is the equation in Grassmann units of the point-curve

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t).$$

Developing $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, $\phi_4(t)$,

$$\begin{aligned} P_t = {} & a_{10}E_1 + a_{20}E_2 + a_{30}E_3 + a_{40}E_4 \\ & + (a_{11}E_1 + a_{21}E_2 + a_{31}E_3 + a_{41}E_4)(t - t_0) \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + (a_{1n}E_1 + a_{2n}E_2 + a_{3n}E_3 + a_{4n}E_4)(t - t_0)^n \\ & + \text{etc.} \end{aligned}$$

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* If we suppose that neither of the lines ξ_0, η_0 touches any branch of the curve in the point ξ_0, η_0, these μ developments arrange themselves in groups of the type

$$\eta - \eta_0 = B_0(\xi - \xi_0) + B_1\omega^{\beta_1}(\xi - \xi_0)^{\frac{\beta_1}{\Delta}} + B_2\omega^{\beta_2}(\xi - \xi_0)^{\frac{\beta_2}{\Delta}} + \text{etc.} \ldots \qquad \text{(a)}$$

where ω is any root of $\omega^{\Delta} = 1$ and the equations differ from one another only in containing different values of ω. $\Sigma\Delta = \mu$, the summation referring to the different groups. Each of the groups (a) defines a branch of the curve with the singularity indices $\Delta - 1$, $\beta_1 - \Delta - 1$. The branch is "linear" if $\Delta = 1$; "superlinear" if $\Delta > 1$; the terminology is Prof. Cayley's. To each branch (a) corresponds a branch of $\phi_i(t)$ of whose singularity indices it will generally be true that the first is $\Delta - 1$ and the second and third depend upon but Δ, β_1, β_2. See in this connection Prof. H. J. S. Smith's memoir on the "Higher Singularities of Plane Curves," Proceedings of London Math. Society, Vol. 6.

Or, if $a_{1j}E_1 + a_{2j}E_2 + a_{3j}E_3 + a_{4j}E_4 = A_j$,

$$P_t = A_0 + A_1(t - t_0) + A_2(t - t_0)^2 + \ldots + A_n(t - t_0)^n + \text{etc.} \ldots \qquad (1)$$

Reciprocally, if u_1, u_2, u_3, u_4 be the coordinates of a plane-element, the equations

$$u_1 : u_2 : u_3 : u_4 = \psi_1(t) : \psi_2(t) : \psi_3(t) : \psi_4(t)$$

yield the Grassmann equation of the plane-curve $\psi_i(t)$. It is

$$\varepsilon_t = \alpha_0 + \alpha_1(t - t_0) + \alpha_2(t - t_0)^2 + \ldots + \alpha_n(t - t_0)^n + \text{etc.} \ldots \qquad (2)$$

The coefficients α are pieces of fixed planes.

§ 2.

By Taylor's theorem, within the circle of convergence about t_0,

$$P_{t_1 + \delta} = P_{t_1} + P'_{t_1}\delta + P''_{t_1}\frac{\delta^2}{2!} + \ldots + P^n_{t_1}\frac{\delta^n}{n!}{}^{*} + \text{etc.}$$

I define the point P_{t_1} as *singular in the first class to the* $\varkappa_1$ *order* when P_{t_1} is congruent with each of the differential points P'_{t_1}, P''_{t_1}, ... $P^{\kappa_1}_{t_1}$, but not with the point $P^{\kappa_1+1}_{t_1}$; or when $[P_{t_1}] \equiv P'_{t_1}, P''_{t_1}, \ldots P^{\kappa_1}_{t_1}, \not\equiv P^{\kappa_1+1}_{t_1}$.

I define a point P_{t_1} whose first class singularity is $\varkappa_1$, as *singular in the second class to the* $\varkappa_2$ *order*, when all the differential points $P^{\kappa_1+2}_{t_1}$, $P^{\kappa_1+3}_{t_1}$, ... $P^{\kappa_1+\kappa_2+1}_{t_1}$ lie in the same straight line with P_{t_1} and $P^{\kappa_1+1}_{t_1}$, but the point $P^{\kappa_1+\kappa_2+2}_{t_1}$ lies off this line; or when

$$[P_{t_1}, P^{\kappa_1+1}_{t_1}] \equiv P^{\kappa_1+2}_{t_1}, P^{\kappa_1+3}_{t_1}, \ldots P^{\kappa_1+\kappa_2+1}_{t_1}, \not\equiv P^{\kappa_1+\kappa_2+2}_{t_1}.$$

A point P_{t_1} whose first and second class singularities are of the orders $\varkappa_1$ and $\varkappa_2$ respectively, I define as *singular in the third class to the* $\varkappa_3$ *order*, when all the differential points $P^{\kappa_1+\kappa_2+3}_{t_1}$, $P^{\kappa_1+\kappa_2+4}_{t_1}$, ... $P^{\kappa_1+\kappa_2+\kappa_3+2}_{t_1}$ lie in the same plane with P_{t_1}, $P^{\kappa_1+1}_{t_1}$, $P^{\kappa_1+\kappa_2+2}_{t_1}$, but the point $P^{\kappa_1+\kappa_2+\kappa_3+3}_{t_1}$ lies without this plane; or when $[P_{t_1}, P^{\kappa_1+1}_{t_1}, P^{\kappa_1+\kappa_2+2}_{t_1}] \equiv P^{\kappa_1+\kappa_2+3}_{t_1}, P^{\kappa_1+\kappa_2+4}_{t_1}, \ldots P^{\kappa_1+\kappa_2+\kappa_3+2}_{t_1} \not\equiv P^{\kappa_1+\kappa_2+\kappa_3+3}_{t_1}$.

As all points of space are linearly derivable from any four among which no linear relation exists, with the three classes just enumerated we exhaust the singularity-defining relations of the points P_{t_1}, P'_{t_1}, P''_{t_1}, etc.

1. *The First Class Singularity.* The infinitesmal change of position of a curve point P_t corresponding to an infinitesmal change δ in the value of t is of the same order as δ if $\varkappa_1$, the index of the first class singularity of P_t, is zero; but if $\varkappa_1$ have any other value than zero, this infinitesmal change of position of P_t is of a higher order than δ, being indeed of the $\varkappa_1 + 1$ order when δ is of the first order. It is supposed of course that t is made to vary continuously.

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* Here, as in what follows, I indicate the order of a derived function by a simple exponent.

For the progressive product, $P_t P_{t+\delta}$ measures the order of the distance of the points P_t, $P_{t+\delta}$ apart; and $P_t P_{t+\delta} = P_t P_t^{\kappa_1+1} \cdot \frac{\delta^{\kappa_1+1}}{(\varkappa_1+1)!} +$ etc., since $P_t^{\kappa_1+1}$ is the first differential point which is not congruent with P_t. Hence, *a point which is singular in the first class is stationary, and the index $\varkappa_1$ measures to what degree.*

2. *The Second Class Singularity.* Let P_{t_2}, P_{t_3} be any two points on the curve P_t, and let them be made, independently of each other, to approach P_{t_1}; the line $P_{t_2}P_{t_3}$, in its limiting position, is, by the ordinary definition, the *tangent* to the curve at the point P_{t_1}.

If $t_2 = t_1 + \delta . \rho_2$, $t_3 = t_1 + \delta . \rho_3$; by Taylor's theorem

$$P_{t_2} = P_{t_1} + P'_{t_1}\delta . \rho_2 + P''_{t_1}\frac{(\delta . \rho_2)^2}{2\,!} + \ldots P_{t_1}^{\kappa_1+1}\frac{(\delta . \rho_2)^{\kappa_1+1}}{(\varkappa_1+1)\,!} + \text{etc.} \ldots$$

$$P_{t_3} = P_{t_1} + P'_{t_1}\delta . \rho_3 + P''_{t_1}\frac{(\delta . \rho_3)^2}{2\,!} + \ldots P_{t_1}^{\kappa_1+1}\frac{(\delta . \rho_3)^{\kappa_1+1}}{(\varkappa_1+1)\,!} + \text{etc.} \ldots$$

Therefore,

$$P_{t_2}P_{t_3} = \begin{vmatrix} 1 & \dfrac{\rho_2^{\kappa_1+1}}{(\varkappa_1+1)\,!} \\ 1 & \dfrac{\rho_3^{\kappa_1+1}}{(\varkappa_1+1)\,!} \end{vmatrix} \delta^{\kappa_1+1} . P_{t_1}P_{t_1}^{\kappa_1+1} + \ldots,$$

so that the tangent at P_{t_1}, or $\underset{\delta \doteq 0}{\text{limit}}\left[\frac{P_{t_2}P_{t_3}}{\delta^{\kappa_1+1}}\right]$, is congruent with $P_{t_1}P_{t_1}^{\kappa_1+1}$.

The tangent to a curve at any point P is the line determined by P and the first of the differential points P', P'', . . . which is not congruent with P; it is the line PP^{κ_1+1}, if $\varkappa_1$ be the index of the first class singularity of the point.

Now the product of the point $P_{t+\delta}$ by $P_t P_t^{\kappa_1+1}$ measures the order of its distance from the tangent; for the product $P_t P_t^{\kappa_1+1} P_{t+\delta}$ is of the same order as the triangle $P_t P_t^{\kappa_1+1} P_{t+\delta}$, and this of the same order as its altitude, the distance of $P_{t+\delta}$ from $P_t P_t^{\kappa_1+1}$. But $P_t P_t^{\kappa_1+1} P_{t+\delta} = P_t P_t^{\kappa_1+1} P_t^{\kappa_1+\kappa_2+2}\frac{\delta^{\kappa_1+\kappa_2+2}}{(\varkappa_1+\varkappa_2+2)\,!} +$ etc., where $P_t^{\kappa_1+\kappa_2+2}$ is the first differential point which lies off the line $P_t P_t^{\kappa_1+1}$; so that $P_{t+\delta}$ is distant from the tangent at P_t by an infinitesmal of the $\varkappa_1 + \varkappa_2 + 2$ order.

At a point of the first and second class singularity $\varkappa_1$, $\varkappa_2$, a curve has with its tangent a contact of the $\varkappa_1 + \varkappa_2 + 1$ order; in addition to the $\varkappa_1$ consecutive points which coalesce with P there lie still other $\varkappa_2 + 1$ consecutive points on the tangent through P.

3. *The Third Class Singularity.* If P_{t_2}, P_{t_3}, P_{t_4} be three curve points and these points be made independently of each other to approach the point P_t, the plane $P_{t_2}P_{t_3}P_{t_4}$ in its limiting position is the *osculating plane* to the curve at P_t.

By a simple extension of the proof in the last paragraph $\underset{\delta \doteq 0}{\text{limit}} \left[\frac{P_{t_2}P_{t_3}P_{t_4}}{\delta^{2\kappa_1+\kappa_2+3}}\right]$ is congruent with $P_tP_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2}$; or

The osculating plane to a curve at any point P is that determined by P, P^{κ_1+1}, the first differential point not congruent with P, and $P^{\kappa_1+\kappa_2+2}$, the first differential point off the line PP^{κ_1+1}.

The progressive product $P_tP_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2}P_{t+\delta}$ measures the order of the distance of $P_{t+\delta}$ from the osculating plane at P_t; but $P_tP_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2}P_{t+\delta}$ $= P_tP_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2}P_t^{\kappa_1+\kappa_2+\kappa_3+3} \dfrac{\delta^{\kappa_1+\kappa_2+\kappa_3+3}}{(\varkappa_1+\varkappa_2+\varkappa_3+3)\,!} +$ etc., since $P^{\kappa_1+\kappa_2+\kappa_3+3}$ is the first differential point not linearly derivable from P, P^{κ_1+1}, $P^{\kappa_1+\kappa_2+2}$. Hence $P_{t+\delta}$ is distant from the osculating plane at P_t by an infinitesmal of the $\varkappa_1+\varkappa_2+\varkappa_3+3$ order.

A third class singularity in P of the order $\varkappa_3$ adds $\varkappa_3$ to the order of contact of the curve with the osculating plane at P; the order of contact at a point whose three singularity indices are $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, being $\varkappa_1+\varkappa_2+\varkappa_3+1$. The number of consecutive points in the osculating plane is $\varkappa_1+\varkappa_2+\varkappa_3+3$.

§ 3.

By Taylor's theorem,

$$\varepsilon_{t+\delta} = \varepsilon_t + \varepsilon_t'\delta + \varepsilon_t''\frac{\delta^2}{2\,!} + \ldots \varepsilon_t^n \frac{\delta^n}{n\,!} + \text{etc.}$$

The plane element ε_t I define as *singular in the first class of plane singularity, and to the μ_1 order*, when $[\varepsilon_t] \equiv \varepsilon_t', \varepsilon_t'', \ldots \varepsilon_t^{\mu_1} \not\equiv \varepsilon_t^{\mu_1+1}$.

The plane ε_t, whose first class singularity is of the order μ_1, I define as *singular in the second class to the μ_2 order* when

$$[\varepsilon_t, \varepsilon_t^{\mu_1+1}] \equiv \varepsilon_t^{\mu_1+2}, \varepsilon_t^{\mu_1+3}, \ldots \varepsilon_t^{\mu_1+\mu_2+1}, \not\equiv \varepsilon_t^{\mu_1+\mu_2+2}.$$

The planes $\varepsilon_t^{\mu_1+2}, \ldots \varepsilon_t^{\mu_1+\mu_2+1}$ pass through the intersection of ε_t and $\varepsilon_t^{\mu_1+1}$, but $\varepsilon_t^{\mu_1+\mu_2+2}$ does not pass through this line.

The plane ε_t, of which the first and second class singularities are of the orders μ_1 and μ_2 respectively, I define as *singular in the third class to the order μ_3*, when $[\varepsilon_t, \varepsilon_t^{\mu_1+1}, \varepsilon_t^{\mu_1+\mu_2+2}] \equiv \varepsilon_t^{\mu_1+\mu_2+3}, \varepsilon_t^{\mu_1+\mu_2+4}, \ldots \varepsilon_t^{\mu_1+\mu_2+\mu_3+2}, \not\equiv \varepsilon_t^{\mu_1+\mu_2+\mu_3+3}$.

All the differential planes preceding $\varepsilon_t^{\mu_1+\mu_2+\mu_3+3}$ pass through the point in which ε_t, $\varepsilon_t^{\mu_1+1}$, $\varepsilon_t^{\mu_1+\mu_2+2}$ intersect, but $\varepsilon_t^{\mu_1+\mu_2+\mu_3+3}$ does not pass through this point.

With these three classes the singularity-defining relations among the coefficients of $\varepsilon_{t+\delta}$ are exhausted; for the space of planes is a region of but the fourth order, so that any plane in space is linearly derivable from any four between which no numerical relation exists.

1. *The First Class Singularity. A plane which is singular in the first class is stationary, and the index μ_1 measures to what degree.*

For the order of the (regressive) product of two planes of finite weight is the same as the order of the angle which they make with each other. If ABC be a triangle of one of the planes, its area the weight of the plane, and ABD a triangle of the other plane, its area the weight of its plane, the product of the planes is $[ABC.ABD] = [ABCD]\,AB$; $[ABCD]$ is scalar, being three times the volume of the tetraedron $ABCD$, and its order is that of the angle between ABC and ABD. But the order of the product $\varepsilon_t\varepsilon_{t+\delta}$, or of $\varepsilon_t\varepsilon_t^{\mu_1+1}\dfrac{\delta^{\mu_1+1}}{(\mu_1+1)!}$ + etc., is $\mu_1 + 1$.

2. *The Second Class Singularity.* The tangent line to the curve ε_t in the element ε_{t_1} is the line of intersection of any two elements ε_{t_2} and ε_{t_3} in what is its limiting position when ε_{t_2} and ε_{t_3} are made, independently of each other, to approach ε_{t_1}. By developing ε_{t_2} and ε_{t_3} and taking their regressive product, it is easily proved that the tangent line to a plane-curve in any plane ε is the line determined by ε and the first of the differential planes ε', ε'', . . . which is not congruent with ε: it is the line $\varepsilon\varepsilon^{\mu_1+1}$ if μ_1 be the index of the first class singularity of ε.

At a plane of the first and second class singularity μ_1, μ_2, a plane-curve has with its tangent a "contact" of the $\mu_1 + \mu_2 + 1$ order: in addition to the μ_1 consecutive planes which coincide with ε, there pass still other $\mu_2 + 1$ through the tangent in ε.

For the order of the angle which $\varepsilon_{t+\delta}$ makes with $\varepsilon\varepsilon^{\mu_1+1}$ is the order of the product $\varepsilon_{t+\delta}\varepsilon\varepsilon^{\mu_1+1}$ or $\mu_1 + \mu_2 + 2$.

3. *The Third Class Singularity.* The point of osculation of an element ε of a plane-curve being defined as the limiting position of the point of intersection of three planes which are made independently of one another to approach ε, it can be proved that it is the point determined by ε, ε^{μ_1+1}, $\varepsilon^{\mu_1+\mu_2+2}$, μ_1 and μ_2 being the first and second class singularity indices of ε.

A third class singularity in ε *of the order* μ_3 *adds* μ_3 *to the order of "contact" of the curve with the point of osculation of* ε; *the order of contact in a plane whose three singularity indices are* μ_1, μ_2, μ_3, *being* $\mu_1 + \mu_2 + \mu_3 + 1$. *The number of consecutive planes through the point of osculation is* $\mu_1 + \mu_2 + \mu_3 + 3$.

For the order of the distance of $\varepsilon_{t+\delta}$ from $\varepsilon\varepsilon^{\mu_1+1}\varepsilon^{\mu_1+\mu_2+2}$ is $\mu_1 + \mu_2 + \mu_3 + 3$.

§ 4.

The curve point P and its osculating plane ε are corresponding elements of a point-curve and a plane-curve. We are to prove that if

$\varkappa_1, \varkappa_2, \varkappa_3$ *be the singularity indices of* P,
$\varkappa_3, \varkappa_2, \varkappa_1$ *are the singularity indices of* ε.

And reciprocally, if

μ_1, μ_2, μ_3 *be the singularity indices of* ε_1,
μ_3, μ_2, μ_1 *are the singularity indices of* P_1.

By hypothesis and the theorem on pages 161 and 162,

$$\varepsilon = PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}.$$

It must be proved that

1. $[\varepsilon] \equiv \varepsilon', \varepsilon'', \ldots \varepsilon^{\kappa_3}, \not\equiv \varepsilon^{\kappa_3+1}.$
2. $[\varepsilon, \varepsilon^{\kappa_3+1}] \equiv \varepsilon^{\kappa_3+2}, \ldots \varepsilon^{\kappa_3+\kappa_2+1} \not\equiv \varepsilon^{\kappa_3+\kappa_2+2}.$
3. $[\varepsilon, \varepsilon^{\kappa_3+1}, \varepsilon^{\kappa_3+\kappa_2+2}] \equiv \varepsilon^{\kappa_3+\kappa_2+3}, \ldots \varepsilon^{\kappa_3+\kappa_2+\kappa_1+2} \not\equiv \varepsilon^{\kappa_3+\kappa_2+\kappa_1+3}.$

1. In the series of planes into which $\varepsilon^j = [PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}]^j$ can be developed the highest index which any point factor can have is $\varkappa_1 + \varkappa_2 + 2 + j$. So long therefore as $j <, = \varkappa_3$, ε^j will be congruent with ε. But if $j = \varkappa_3 + 1$ the term

$$PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+\kappa_3+3}$$

enters into the expansion of ε^j, and this cannot be congruent with ε since $P^{\kappa_1+\kappa_2+\kappa_3+3}$ lies without ε by hypothesis.

2. $$\varepsilon^{\kappa_3+1+j} = [PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}]^{\kappa_3+1+j}$$

which equals, numerical coefficients being omitted,

$$[PP^{\kappa_1+1}]^{\kappa_3+1+j}P^{\kappa_1+\kappa_2+2} + \ldots [PP^{\kappa_1+1}]^{\kappa_3+1+j-\kappa_3}P^{\kappa_1+\kappa_2+\kappa_3+2} \qquad \text{(a)}$$
$$+ [PP^{\kappa_1+1}]^{j}P^{\kappa_1+\kappa_2+\kappa_3+3} + \ldots [PP^{\kappa_1+1}]P^{\kappa_1+\kappa_2+\kappa_3+3+j}. \qquad \text{(b)}$$

Now for every value of j, from 1 through $\varkappa_2$, all the products (a) are of points whose indices are $<, = \varkappa_1 + \varkappa_2 + \varkappa_3 + 2$; so that for every j which is $<, = \varkappa_2$, the planes (a) are congruent with ε.

And again, under the same hypothesis with reference to j, every factor $[PP^{\kappa_1+1}]^l$ occurring in the terms of (b) develops into a sum of lines which are congruent with PP^{κ_1+1}; so that (b) itself is a sum of planes all of which pass through PP^{κ_1+1}, the intersection of ε with ε^{κ_3+1}.

If, on the other hand, $j = \varkappa_2 + 1$, there occurs in (a), in addition to terms which are congruent with ε, the term $PP^{\kappa_1+\kappa_2+\kappa_3+3}P^{\kappa_1+\kappa_2+2}$ with the numerical coefficient 1; and in (b), in addition to planes through the intersection of ε and ε^{κ_3+1}, the term

$$\frac{(\varkappa_3+\varkappa_2+1)!}{(\varkappa_2+1)!\,(\varkappa_3+1)!}\,PP^{\kappa_1+\kappa_2+2}P^{\kappa_1+\kappa_2+\kappa_3+3}.$$

Now, since $P^{\kappa_1+\kappa_2+\kappa_3+3}$ is without ε and $P^{\kappa_1+\kappa_2+2}$ off PP^{κ_1+1}, $PP^{\kappa_1+\kappa_2+\kappa_3+3}P^{\kappa_1+\kappa_2+2}$ cannot pass through $\varepsilon\varepsilon^{\kappa_3+1}$; nor can the term in any case disappear, since no integral positive values of $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ can be found capable of solving the equation $\dfrac{(\varkappa_3+\varkappa_2+1)!}{(\varkappa_2+1)!\,(\varkappa_3+1)!} = 1$.

$$
\begin{aligned}
3.\quad \varepsilon^{\kappa_3+\kappa_2+2+j} &= \left[PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+\kappa_2+2+j} \\
&\left.\begin{aligned} &= P^{\kappa_3+\kappa_2+2+j}\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}+\ldots\right. \\ &+ P^{\kappa_3+\kappa_2+2+j-\kappa_3}\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3} \end{aligned}\right\} \qquad \text{(a)} \\
&\left.\begin{aligned} &+ P^{\kappa_2+1+j}\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+1}+\ldots \\ &+ P^{\kappa_2+1+j-\kappa_2}\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+1+\kappa_2} \end{aligned}\right\} \qquad \text{(b)} \\
&\left.\begin{aligned} &+ P^{j}\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+\kappa_2+2}+\ldots \\ &+ P\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+\kappa_2+2+j}. \end{aligned}\right\} \qquad \text{(c)}
\end{aligned}
$$

For every value of j, from 0 through $\varkappa_1$—

1.) The sum of the terms (a) is a plane which is congruent with ε, for the highest index which any point factor can bear is $\varkappa_3 + \varkappa_2 + \varkappa_1 + 2$.

2.) The sum of the terms (b) is a plane which passes through the intersection of ε and ε^{κ_1+1}. For each of these terms is the product of a point of the line PP^{κ_1+1} into a line $\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+1+l}$, $l = 0, 1, 2, \ldots \varkappa_2$. But

$$
\begin{aligned}
\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+1+l} &= P^{\kappa_1+1}P^{\kappa_1+\kappa_2+\kappa_3+3+l}+\ldots+P^{\kappa_1+\kappa_2+1}P^{\kappa_1+\kappa_3+3+l} \quad (\mathrm{a}')_l \\
&+ P^{\kappa_1+\kappa_2+2}P^{\kappa_1+\kappa_3+2+l}+\ldots+P^{\kappa_1+\kappa_3+2+l}P^{\kappa_1+\kappa_2+2}, \quad (b')_l
\end{aligned}
$$

where all the lines $(\mathrm{a}')_l$ meet PP^{κ_1+1} and all the lines $(\mathrm{b}')_l$ lie in the plane ε. It follows, therefore that (b) which is

$$= \sum_{l=0}^{l=\kappa_2} P^{\kappa_2+1+j-l}\left[P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\right]^{\kappa_3+1+l}$$

is a plane through PP^{κ_1+1}, *i. e.* through $\varepsilon\varepsilon^{\kappa_3+1}$.

3.) Each of the planes in the sum (c) passes through P, since

$$P \equiv P', P'', \dots P^{\kappa_1}$$

by hypothesis. Each of the planes $\varepsilon^{\kappa_3+\kappa_2+2+j}$, therefore, where j takes in succession the values $0, 1, \dots \varkappa_1$ passes through the point P.

On the other hand, the plane $\varepsilon^{\kappa_3+\kappa_2+\kappa_1+3}$ cannot pass through P. For in its development occurs, in addition to planes through P, the term

$$p^{\kappa_3+\kappa_2+\kappa_1+3}p^{\kappa_1+1}p^{\kappa_1+\kappa_2+2}$$

with the numerical coefficient

$$1-\frac{(\varkappa_3+\varkappa_2+\varkappa_1+3)!}{(\varkappa_2+\varkappa_1+2)!\,(\varkappa_3+1)!}-\frac{(\varkappa_3+\varkappa_2+\varkappa_1+3)!}{(\varkappa_1+1)!\,(\varkappa_3+\varkappa_2+2)!}+\frac{\varkappa_3+\varkappa_2+\varkappa_1+3}{(\varkappa_1+1)!\,\varkappa_2+1)!\,(\varkappa_3+1)!}$$

and only this term, and this term can in no case disappear.

§ 5.[*]

At a curve point P the generating point continues the direction of its motion along the tangent line at P, if $\varkappa_1$, the index of the first class singularity of P be even, but reverses the direction of this motion if $\varkappa_1$ be odd.	*At the element ε of a plane curve the enveloping element maintains the sense of its turning about the tangent line in ε, if μ_1, the index of the first class singularity of ε be even, but reverses the sense of this turning if μ_1 be odd.*

The generating point takes in succession the positions $P_{t-\delta}$, P_t, $P_{t+\delta}$. We wish to determine whether the projections of the points $P_{t-\delta}$, $P_{t+\delta}$ on the tangent line lie to the same side of P_t or to opposite sides of it. All depends on whether the two products $P_tP_{t+\delta}$, $P_tP_{t-\delta}$ have the same or opposite signs; but, on the hypothesis that the index of the first class singularity of P_t is $\varkappa_1$, these products are $P_tP_t^{\kappa_1+1}\dfrac{\delta^{\kappa_1+1}}{(\varkappa_1+1)!}$ and $P_tP_t^{\kappa_1+1}\dfrac{(-\delta)^{\kappa_1+1}}{(\varkappa_1+1)!}$, respectively, and so have the same sign when $\varkappa_1$ is odd, opposite signs when $\varkappa_1$ is even.

At the point P the projection of the generating point on the osculating plane at P crosses the tangent if $\varkappa_1+\varkappa_2$, the sum of the indices of the first and second class singularity of P, be an odd number, but remains to the same side of this line if $\varkappa_1+\varkappa_2$ be an even number.	*At the element ε of a plane-curve the line in which the enveloping element cuts ε, maintains the sense of its turning in ε if $\mu_1+\mu_2$, the sum of the indices of the first and second class singularity of ε be an odd number, but reverses the sense of this turning if $\mu_1+\mu_2$ be an even number.*

---

[*] Here and wherever else the *form* of the curve is discussed, it is supposed of course that the coefficients of the series as well as t itself are real.

The proof is in principle the same as that given for the preceding theorem. We take the product of the tangent PP^{κ_1+1} into the point $P_{t+\delta}$ and into the point $P_{t-\delta}$; on the hypothesis that P_t has the singularity indices $\varkappa_1$, $\varkappa_2$, these products are

$$PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\frac{\delta^{\kappa_1+\kappa_2+2}}{(\varkappa_1+\varkappa_2+2)!} \quad\text{and}\quad PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}\frac{(-\delta)^{\kappa_1+\kappa_2+2}}{(\varkappa_1+\varkappa_2+2)!},$$

and have the same or opposite signs according as $\varkappa_1+\varkappa_2$ is even or odd.

At the point P the generating point crosses the osculating plane if $\varkappa_1+\varkappa_2+\varkappa_3$, the sum of the three singularity indices of P, be an even number, but remains to the same side of it if $\varkappa_1+\varkappa_2+\varkappa_3$ be an odd number.

At the element ε of a plane-curve the enveloping element crosses the point of osculation if $\mu_1+\mu_2+\mu_3$, the sum of the three singularity indices of ε be an even number, but does not cross it if $\mu_1+\mu_2+\mu_3$ be an odd number.

The product of $P_tP_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2}$ into $P_{t+\delta}$ is

$$P_tP_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2}P_t^{\kappa_1+\kappa_2+\kappa_3+3}\frac{\delta^{\kappa_1+\kappa_2+\kappa_3+3}}{(\varkappa_1+\varkappa_2+\varkappa_3+3)!},$$

and its product into $P_{t-\delta}$ is $P_tP_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2}P_t^{\kappa_1+\kappa_2+\kappa_3+3}\dfrac{(-\delta)^{\kappa_1+\kappa_2+\kappa_3+3}}{(\varkappa_1+\varkappa_2+\varkappa_3+3)!}$; these products have opposite signs if $\varkappa_1+\varkappa_2+\varkappa_3$ be even, the same signs if $\varkappa_1+\varkappa_2+\varkappa_3$ be odd.

If a point-curve be projected into an axial pencil or a plane-curve into a range of points, to the singularities of the originals will correspond singularities in the projections, the singularities of the projections, however, depending also on the position of the line which is the base of the pencil or range. With reference to the classes of singularity possible to the two systems into which we project—the same considerations which prove three sorts of singularity to be possible to an element of a point- or plane-curve of double curvature prove that but one sort is possible to the element of an axial pencil or a range of points; in these systems there are but *two* linearly independent elements and the one source of singularity is that elements should coincide, which, in the general case, are distinct.

Let P be an element of a point-curve of double curvature with the singularity indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, let l be the tangent line and ε the osculating plane to the curve at P, and a the base of the axial pencil into which the curve is projected; also, let ε_1 be an element of a plane-curve, μ_1, μ_2, μ_3 its singularity indices, l_1 its tangent line, P_1 its point of osculation, and a_1 the base of the range into which it is projected. Then

The singularity index of that element of the axial pencil a which is the projection of P, is $\varkappa_1$, provided the point $a\varepsilon$ does not lie in l, (1); $\varkappa_1 + \varkappa_2 + 1$ when $a\varepsilon$ is in l but is not P, (2); $\varkappa_2$ when $a\varepsilon$ is congruent with P, (3); $\varkappa_1 + \varkappa_2 + \varkappa_3 + 2$ when a is in ε but does not pass through P, (4); $\varkappa_3 + \varkappa_{2+1}$ when a is in ε and through P, (5); and $\varkappa_3$ when a is itself l, (6).

The singularity index of that point of the range a_1 which is the projection of ε_1 is μ_1, provided the plane a_1P_1 does not pass through l_1 (1); is $\mu_1 + \mu_2 + 1$ when a_1P_1 passes through l_1 but does not coincide with ε_1, (2); μ_2 when a_1P_1 is congruent with ε_1, (3); $\mu_1 + \mu_2 + \mu_3 + 2$ when a_1 passes through P_1 but does not lie in ε_1, (4); $\mu_3 + \mu_2 + 1$ when a_1 is in ε_1 and contains P_1, (5); and μ_3 when a_1 is itself l_1, (6).

For if $P_{t+\delta}$ be a curve point consecutive to P_t, the projections of P_t and $P_{t+\delta}$, *i. e.* aP_t and $aP_{t+\delta}$, are consecutive elements of the axial pencil a. In case (1), *i. e.* when a does not meet l, $aP_{t+\delta} = aP + aP'.\delta +$ etc.; here $aP \equiv aP'$, aP'', … aP^{κ_1}, $\not\equiv aP^{\kappa_1+1}$, and the singularity index of aP is $\varkappa_1$. In case (2)$a = P_0(\alpha P_t + \beta P_t^{\kappa_1+1})$, P_0 being a point without ε; hence $aP_{t+\delta} = P_0(\alpha P_t + \beta P_t^{\kappa_1+1})(P_t + P'\delta +$ etc.) and all the planes of this expansion through

$$P_0(\alpha P_t + \beta P_t^{\kappa_1+1})\frac{P^{\kappa_1+\kappa_2+1}\delta^{\kappa_1+\kappa_2+1}}{(\varkappa_1+\varkappa_2+1)!}$$

are congruent with $P_0(\alpha P_t + \beta P_t^{\kappa_1+1})P_t$, but not the plane next following; so that the singularity index of aP is $\varkappa_1 + \varkappa_2 + 1$. When a passes through P, (3), it is equal to P_0P_t, P_0 having the same meaning as before. And

$$aP_{t+\delta} = P_0P_t(P_t + P_t'\delta + \text{etc.}) = P_0P_tP_t^{\kappa_1+1}\frac{\delta^{\kappa_1+1}}{(\varkappa_1+1)!} + \text{etc., or } \frac{aP_{t+\delta}}{\delta^{\kappa_1+1}} = \frac{P_0P_tP_t^{\kappa_1+1}}{(\varkappa_1+1)!}$$

$+$ etc. Of the terms of this expansion the first $(\varkappa_1 + \varkappa_2 + 1) - (\varkappa_1 + 1)$ are congruent, so that the singularity index of aP is $\varkappa_2$. The other cases admit of similar treatment. It seems necessary to remark only that in (4) $a = \alpha P_tP_t^{\kappa_1+1} + \beta P_t^{\kappa_1+1}P_t^{\kappa_1+\kappa_2+2} + \gamma PP^{\kappa_1+\kappa_2+2}$, in (5) $a = \alpha P_tP_t^{\kappa_1+1} + \beta P_tP_t^{\kappa_1+\kappa_2+2}$, and in (6) $a \equiv PP^{\kappa_1+1}$.

The demonstrations of the theorems in the right-hand column are simply the reciprocals of those just given. We have but to substitute the regressive products of planes for the progressive products of points.

Evidently the singularity of P is determined when the singularities of its projections in any three of the axial pencils (1), (2), … (6) are known; and the like of ε_1.

The indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ determine whether the projections of P in the several pencils a are "Rückkehrelemente"[*] or not; for the generating element of any of these systems maintains or reverses the sense of its turning about a as it reaches any fixed element, according as the singularity index of that element is even or odd; which in the case before us depends solely on the values of $\varkappa_1$, $\varkappa_2$, $\varkappa_3$. A similar relation holds between the numbers μ_1, μ_2, μ_3 and the motion of the generating element of each of the several ranges into which we project ε_1.

Besides projecting into one dimensional system, we may project both of our curves into two dimensional systems also; the point-curve into a pencil of rays, the plane-curve into the system of lines in a plane. The picture of the point-curve is then a cone, and the picture of the plane-curve a line-curve. Here, again, to singularities in the originals correspond singularities in the projections—the exact character of the correspondence depending on the position of the base of the pencil, or of the field of lines. The classes of singularity possible to an element of a cone or of a line-curve in a plane are two, the number of linearly independent elements in each system being but three. A ray l is singular in the first class to the order λ_1, and in the second class to the order λ_2, when

$$l \equiv l', l'', \ldots l^{\lambda_1} \equiv\!|\!\equiv l^{\lambda_1+1} \text{ and } [l, l^{\lambda_1+1}] \equiv l^{\lambda_1+2}, \ldots l^{\lambda_1+\lambda_2+1}, \equiv\!|\!\equiv l^{\lambda_1+\lambda_2+2}.$$

Let the symbols P, l, ε, P_1, l_1, ε_1 have the same meaning as in the preceding theorem; and let the point-curve P_t be projected into the pencil of rays whose centre is A and the plane-curve ε_t into the field α. Then

The two singularity indices of the ray which is the projection of P are $\varkappa_1$, $\varkappa_2$, when A does not lie in ε; $\varkappa_1$, $\varkappa_2 + \varkappa_3 + 1$, when A lies in ε but not in l; $\varkappa_1 + \varkappa_2 + 1$, $\varkappa_3$ when A lies in l but is not P; $\varkappa_2$, $\varkappa_3$ when A is P.

The two singularity indices[†] *of the line in which ε_1 intersects α are μ_1, μ_2 when α does not pass through P_1; μ_1, $\mu_2 + \mu_3 + 1$ when α passes through P_1 but not through l_1; $\mu_1 + \mu_2 + 1$, μ_3 when α passes through l_1 but is not ε_1; μ_2, μ_3 when α is ε_1.*

For when A does not lie in ε, in the product $AP_{t+\delta}$, or $AP + AP'\delta +$ etc., $AP \equiv AP', \ldots AP^{\kappa_1}, \equiv\!|\!\equiv AP^{\kappa_1+1}$ and $[AP, AP^{\kappa_1+1}] \equiv AP^{\kappa_1+2}, \ldots AP^{\kappa_1+\kappa_2+1}, \equiv\!|\!\equiv AP^{\kappa_1+\kappa_2+2}$. If A lie in ε but not in l, $AP \equiv AP', \ldots AP^{\kappa_1}, \equiv\!|\!\equiv AP^{\kappa_1+1}$ and $[AP, AP^{\kappa_1+1}] \equiv AP^{\kappa_1+2}, \ldots AP^{\kappa_1+\kappa_2+\kappa_3+2}, \equiv\!|\!\equiv AP^{\kappa_1+\kappa_2+\kappa_3+3}$; and the indices are $\varkappa_1$ and $\varkappa_1 + \underline{\varkappa_2 + \varkappa_3 + 2} - \underline{(\varkappa_1 + 1)} = \varkappa_2 + \varkappa_3 + 1$. When A lies in l and is not P, $AP = AP', \ldots AP^{\kappa_1+\kappa_2+1}, \equiv\!|\!\equiv AP^{\kappa_1+\kappa_2+2}$ and $[AP, AP^{\kappa_1+\kappa_2+2}] \equiv AP^{\kappa_1+\kappa_2+3}, \ldots$

---

[*] Conf. von Staudt, Geom. d. Lage, p. 113.

[†] The two singularity indices of the element of a line-curve in a plane are its "inflexional" and ‘ cuspidal" indices respectively.

$AP^{\kappa_1+\kappa_2+\kappa_3+2}$, $\not\equiv AP^{\kappa_1+\kappa_2+\kappa_3+3}$; and the indices are $\varkappa_1+\varkappa_2+1$ and $(\varkappa_1+\varkappa_2+\varkappa_3+2) - (\varkappa_1+\varkappa_2+2) = \varkappa_3$. Finally, when A is P all the terms $AP, \ldots AP^{\kappa_1}$ vanish, so that the value of $AP_{t+\delta}$ expresses itself in a series of which the first term is $AP^{\kappa_1+1}\dfrac{\delta^{\kappa_1+1}}{(\varkappa_1+1)!}$; with this term all those immediately following are congruent, while from it and $AP^{\kappa_1+\kappa_2+2}$ all the terms through $AP^{\kappa_1+\kappa_2+\kappa_3+2}$ are linearly derivable; in this case, therefore, the singularity indices are $\varkappa_2$ and $\varkappa_3$.

The theorems in the right-hand column are the reciprocals of those just demonstrated.

§6.

By a "line-curve," I mean a developable, *i. e.* not any continuous ∞^1 aggregate of the general line elements (Liniensummen) of space—such as the point-curve is of points and the plane-curve of planes—but only aggregates whose elements are the tangents to a point- or plane-curve. The general element of a line-curve δ then is $[PP^{\kappa_1+1}]_t$ or $[\varepsilon\varepsilon^{\kappa_3+1}]_t$. Its singularity finds expression in linear relations among the coefficients in the expansion of $[PP^{\kappa_1+1}]_{t+\delta}$. As there are in space but six linearly independent line elements, the possible number of these singularity-defining relations among the coefficients is five. We are to investigate to what extent they are consequent on the singularity of P_t and to what extent they are independent.

The general term of the expansion of $[PP^{\kappa_1+1}]_{t+\delta}$ is $\dfrac{\delta^j}{j!}[PP^{\kappa_1+1}]^j$. Omitting numerical coefficients

$$
\begin{aligned}
[PP^{\kappa_1+1}]^j = {} & PP^{\kappa_1+1+j} + P'P^{\kappa_1+1+j-1} + \ldots P^{\kappa_1}P^{\kappa_1+1+j-\kappa_1} && \text{(a)}\\
& + P^{\kappa_1+1}P^j + P^{\kappa_1+2}P^{j-1} + \ldots P^{\kappa_1+\kappa_2+1}P^{j-\kappa_2} && \text{(b)}\\
& \left.\begin{aligned} & + P^{\kappa_1+\kappa_2+2}P^{j-(\kappa_2+1)} + P^{\kappa_1+\kappa_2+3}P^{j-(\kappa_2+2)} \\ & \qquad + \ldots P^{\kappa_1+\kappa_2+\kappa_3+2}P^{j-(\kappa_2+\kappa_3+1)} \end{aligned}\right\} && \text{(c)}\\
& \left.\begin{aligned} & + P^{\kappa_1+\kappa_2+\kappa_3+3}P^{j-(\kappa_2+\kappa_3+2)} + P^{\kappa_1+\kappa_2+\kappa_3+4}P^{j-(\kappa_2+\kappa_3+3)} \\ & \qquad + \ldots P^{\kappa_1+2\kappa_2+\kappa_3+3}P^{j-(2\kappa_2+\kappa_3+2)} \end{aligned}\right\} && \text{(d)}\\
& + P^{\kappa_1+2\kappa_2+\kappa_3+4}P^{j-(2\kappa_2+\kappa_3+3)} + \ldots P^jP^{\kappa_1+1}. && \text{(e)}
\end{aligned}
$$

1. It will be noticed that the coefficients $[PP^{\kappa_1+1}]'$, $[PP^{\kappa_1+1}]'' \ldots [PP^{\kappa_1+1}]^{\kappa_2}$ are congruent with PP^{κ_1+1}; but not so the next following coefficient. For so long as j does not exceed $\varkappa_2$, $[PP^{\kappa_1+1}]^j$ is a sum of products of points which all lie on the line PP^{κ_1+1}; $[PP^{\kappa_1+1}]^{\kappa_2+1}$, on the other hand, contains in its expansion $PP^{\kappa_1+\kappa_2+2}$.

2. The coefficients to $[PP^{\kappa_1+1}]^{\kappa_2+\kappa_1+1}$ inclusive belong to the pencil of rays through P; but not so the coefficient next following. For so long as j does not

exceed $\varkappa_2 + \varkappa_1 + 1$, the sum into which $[PP^{\kappa_1+1}]^j$ expands terminates in the row (b); the lines (a) all pass through P and the lines (b) coincide with PP^{κ_1+1}: $[PP^{\kappa_1+1}]^{\kappa_2+\kappa_1+2}$, on the other hand, contains in its expansion $P^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}$, which cannot pass through P.

3. The coefficients to $[PP^{\kappa_1+1}]^{\kappa_2+\kappa_3+1}$ inclusive belong to the field of lines $PP^{\kappa_1+1}P^{\kappa_1+\kappa_2+2}$, *i. e.* ε; but the next coefficient lies without this field. For all the points $P, \ldots P^{\kappa_1+\kappa_2+\kappa_3+2}$ lie in ε, but $P^{\kappa_1+\kappa_2+\kappa_3+3}$ without ε.

From 2 and 3 it follows that the coefficients to $[PP^{\kappa_1+1}]^{\kappa_2+\left\{\begin{smallmatrix}\kappa_1\\ \kappa_3\end{smallmatrix}\right\}+1}$ inclusive belong to the flat pencil PP^{κ_1+1}, $PP^{\kappa_1+\kappa_2+2}$; it being understood that that one of the indices $\varkappa_1$, $\varkappa_3$ is to be read which is not the greater.

4. Every coefficient $[PP^{\kappa_1+1}]^{\kappa_1+\kappa_2+\kappa_3+2}$ belongs to one of the two systems, the pencil P or the field ε, or to both of them; but the coefficient $[PP^{\kappa_1+1}]^{\kappa_1+\kappa_2+\kappa_3+3}$ belongs to neither of them. For so long as j does not exceed $\varkappa_1 + \varkappa_2 + \varkappa_3 + 2$, all the terms (a) in the expansion of $[PP^{\kappa_1+1}]^j$ pass through P, and all the terms from $P^{\kappa_1+1}P^j$ on lie in ε. $[PP^{\kappa_1+1}]^{\kappa_1+\kappa_2+\kappa_3+3}$, on the other hand, contains $P^{\kappa_1+1}P^{\kappa_1+\kappa_2+\kappa_3+3}$, which does neither the one nor the other.

5. The coefficients to $[PP^{\kappa_1+1}]^{\kappa_1+2\kappa_2+\kappa_3+3}$ inclusive belong to the linear complex of which the axis is PP^{κ_1+1}; but not the coefficient $[PP^{\kappa_1+1}]^{\kappa_1+2\kappa_2+\kappa_3+4}$. For so long as j does not exceed $\varkappa_1 + 2\varkappa_2 + \varkappa_3 + 3$, the expansion of $[PP^{\kappa_1+1}]^j$ consists only of terms out of (a), (b), (c), (d); and of these the terms (a), (b), (d) contain a point of PP^{κ_1+1} as one of their factors and the terms (c) are the products of points which all lie in the plane ε. On the other hand, the expansion of $[PP^{\kappa_1+1}]^{\kappa_1+2\kappa_2+\kappa_3+4}$ contains the term $P^{\kappa_1+\kappa_2+2}P^{\kappa_1+\kappa_2+\kappa_3+3}$, a line joining a point not on PP^{κ_1+1} with a point not in ε.

Therefore, the defining characteristics (defining because sufficient as well as necessary) of l, the element of the developable l_t corresponding to the point $P(\varkappa_1, \varkappa_2, \varkappa_3)$, are:

I. (1) *l is a line through P.*

(2) *l^{κ_2+1} is also a line through P, but not coincident with l.*

(3) *$l^{\kappa_2+\kappa_1+2}$ (if $\varkappa_1 < \varkappa_3$) is a line in the plane of l and l^{κ_2+1}, but not through P.*

(4) *$l^{\kappa_2+\kappa_3+2}$ is the sum of a line through P and not in the plane of l and l^{κ_2+1}, and a line in the plane of l and l^{κ_2+1} and not through P.*

(5) *$l^{\kappa_1+\kappa_2+\kappa_3+3}$ is an element of the linear complex l which is not resolvable into a line through P and a line in the plane of l and l^{κ_2+1}.*

(6) *$l^{\kappa_1+2\kappa_2+\kappa_3+4}$ is not an element of the linear complex l.*

II. (1) $l \equiv l', l'', \ldots l^{\kappa_2}$.

(2) $[l, l^{\kappa_2+1}] \equiv l^{\kappa_2+2}, l^{\kappa_2+3}, \ldots l^{\kappa_2+\kappa_1+1}$.

(3) $[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}] \equiv l^{\kappa_2+\kappa_1+3}, l^{\kappa_2+\kappa_1+4}, \ldots l^{\kappa_2+\kappa_3+1}$.

(4) $[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}, l^{\kappa_2+\kappa_3+2}] \equiv l^{\kappa_2+\kappa_3+3}, \ldots l^{\kappa_1+\kappa_2+\kappa_3+2}$.

(5) $[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}, l^{\kappa_2+\kappa_3+2}, l^{\kappa_1+\kappa_2+\kappa_3+3}] \equiv l^{\kappa_1+\kappa_2+\kappa_3+4}, \ldots l^{\kappa_1+2\kappa_2+\kappa_3+3}$.

N. B. If $\varkappa_1 > \varkappa_3$, the indices $\varkappa_1$ and $\varkappa_3$ are to change places in both I and II. $l^{\kappa_2+\kappa_3+2}$ is then a line through P, but not in the plane of l and l^{κ_2+1}, and $l^{\kappa_2+\kappa_1+2}$ the sum of two lines, the one through P, the other in the plane of l and l^{κ_2+1}. If $\varkappa_1 = \varkappa_3$, both I (3) and II (3) disappear. The only one of the l^j's which *must* be the general line element, complex, is (4).

We may regard II (1), (2), (3), (4), (5) as the definitions of line singularities of four or five classes with order indices $\varkappa_2$, $\varkappa_1$, etc. They are conditioned solely by the singularity of P.

The possibility of an independent line singularity depends entirely on whether $\varkappa_1$ and $\varkappa_3$ be equal or unequal. For when $\varkappa_1 \neq \varkappa_3$, there are among the coefficients to $l^{\kappa_1+2\kappa_2+\kappa_3+4}$ inclusive six linearly independent lines, and from six linearly independent lines all the lines and complexes of space can be derived; in this case no singularity beyond II (5) is possible. But if $\varkappa_1 = \varkappa_3$, (3) falls away, and there are but five linearly independent lines among the coefficients to $l^{\kappa_1+2\kappa_2+\kappa_3+4}$ inclusive. These determine a linear complex to which a number of the coefficients $l^{\kappa_1+2\kappa_2+\kappa_3+5}$, $l^{\kappa_1+2\kappa_2+\kappa_3+6}$, etc., may belong.

I define an element of a developable as *singular in the line-class to the order λ when $\varkappa_1 = \varkappa_3$, and*

$$[l, l^{\kappa_2+1}, l^{\kappa_2+\kappa_1+2}, l^{\kappa_2+2\kappa_1+3}, l^{2\kappa_2+2\kappa_1+4}] \equiv l^{2\kappa_1+2\kappa_2+5}, \ldots l^{2\kappa_1+2\kappa_2+\lambda+4},$$
$$\not\equiv l^{2\kappa_1+2\kappa_2+\lambda+5}.$$

The geometrical character of the singularities of the developable can be most easily studied by aid of its projection in a point-space of five dimensions. Let $l_1, l_2, \ldots l_6$ be any six linearly independent lines or linear complexes of an S_3, and let the six linearly independent points $P_1, P_2, \ldots P_6$ be their projections in S_5; every line and linear complex of S_3 is then of the form

$$\alpha_1 l_1 + \alpha_2 l_2 + \alpha_3 l_3 + \alpha_4 l_4 + \alpha_5 l_5 + \alpha_6 l_6,$$

and its projection in S_5 is $\alpha_1 P_1 + \alpha_2 P_2 + \ldots \alpha_6 P_6$. To linear relations among elements of the one space correspond linear relations among their projections in the other. Thus a ray of the flat pencil $\alpha_1 l_1 + \alpha_2 l_2$ (l_1 and l_2 being intersecting lines) projects into a point of the range $\alpha_1 P_1 + \alpha_2 P_2$, the rays of a pencil or the lines of a plane in S_3 project into the points of a plane in S_5, the elements of a linear complex in S_3, into the points of an S_4 in S_5. In particular, a developable in S_3 projects into a point-curve in S_5, its singular elements into singular elements of the point-curve with the same singularity indices. Applying now to this point-curve the methods developed in §2 and interpreting the results for the developable, we infer the following theorems: P, ε, l are supposed to be corresponding elements of the point-, plane- and line-curve C, with singularity indices $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, λ.

1. *l is stationary to the degree $\varkappa_2$.*

2. *The developable has in l a contact of the $\varkappa_2 + \varkappa_1 + 1$ order with P; $\varkappa_2 + \varkappa_1 + 2$ consecutive elements of the developable belong to the pencil P.*

3. *The developable has in l a contact of the $\varkappa_2 + \varkappa_3 + 1$ order with ε; $\varkappa_2 + \varkappa_3 + 2$ consecutive elements of the developable belong to the field ε.*

4. *The developable has in l a contact of the $\varkappa_1 + 2\varkappa_2 + \varkappa_3 + 1$ order with the linear complex of which l is the axis; $\varkappa_1 + 2\varkappa_2 + \varkappa_3 + 4$ consecutive elements of the developable belong to this complex.*

5. *In particular, when $\varkappa_1 = \varkappa_3$, the developable has in l a contact of the order $2\varkappa_1 + 2\varkappa_2 + \lambda + 1$ with the linear complex determined by l, l^{κ_2+1}, $l^{\kappa_2+\kappa_1+2}$, $l^{2\kappa_1+\kappa_2+3}$, $l^{2\kappa_1+2\kappa_2+4}$; $2\varkappa_1 + 2\varkappa_2 + \lambda + 5$ consecutive elements of the developable belong to this complex.*

In case $\varkappa_1 = \varkappa_3$, as the generating element of the developable moves away from l it first turns about P in ε, then splits into two lines, one through P and not in ε, the other in ε and not through P, becomes in succession a general element of the complex l and of the complex $(l, \ldots l^{2\kappa_1+2\kappa_2+4})$, and finally a general element of line space. In the other cases, before splitting it becomes a line in ε not through P, or a line through P not in ε; and passes directly from the complex l into unrestricted line space.

Again, at l the generating element

1. *maintains the sense of its turning in ε about P if $\varkappa_2$ be even, reverses it if $\varkappa_2$ be odd.*
2. *in turning about P, crosses ε or not according as $\varkappa_2 + \varkappa_3$ is odd or even.*[*]
3. *in its motion in ε crosses P or not according as $\varkappa_2 + \varkappa_1$ is odd or even.*
4. *moves across l or not according as $\varkappa_1 + \varkappa_3$ is odd or even.*

The intersection of the developable with any plane α is a point-curve in α; and its projection in the pencil of planes through any point A is a cone of this pencil.

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[*] *I. e.* every point on it crosses ε or not, as the case may be.

The two singularity indices of the point in α *which is the trace of* l *are* κ_2, κ_3 *when* α *does not pass through* P; $\kappa_2 + \kappa_1 + 1$, κ_3 *when* α *passes through* P *but not through* l; κ_1, $\kappa_2 + \kappa_3 + 1$ *when* α *passes through* l *but is not* ε; κ_1, κ_2 *when* α *is* ε.

The two singularity indices of the plane through A *which is the projection of* l *are* κ_2, κ_1 *when* A *does not lie in* ε; $\kappa_2 + \kappa_3 + 1$, κ_1 *when* A *lies in* ε *but not in* l; κ_3, $\kappa_2 + \kappa_1 + 1$ *when* A *lies in* l *but is not* P; κ_3, κ_2 *when* A *is* P.

These theorems have been already demonstrated (p. 169). They may also be demonstrated independently by aid of the development $l_{t+\delta} = l + l'\delta +$ etc.

From the Plücker equations connecting the ordinary singularities of the projections of the developable in α and A, Prof. Cayley has deduced equations connecting the ordinary singularities of the developable itself.[*] In the Plücker equations for the trace of the developable in α (α not passing through P) the singularity κ_1, κ_2, κ_3 counts for κ_2 simple cusps and κ_3 simple inflexions; in the Plücker equations for the cone of planes which is the projection of the developable in A (A not in ε), it counts for κ_2 simple inflexions and κ_1 simple cusps. Therefore—

In respect to its influence on the order, class, rank and genus of its curve, the singularity κ_1, κ_2, κ_3 *is equivalent to* κ_1 *points,* κ_2 *lines,* κ_3 *planes, all stationary to the first degree.*

N. B. No account is here taken of the double points, double planes, etc., which enter also into the composition of higher singularities—due to the mutual intersections of the projections of the several partial branches which unite in the singular element. The series with which we operate and which define our singularities represent but single partial branches.[†]

§7.

An analytical point-curve of double curvature being defined in the neighborhood of the point $(x)_0$ by the equations

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t),$$

where

$$\phi_i(t) = a_{i0} + a_{i1}(t - t_0) + \ldots a_{in}(t - t_0)^n + \text{etc.}$$

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[*] *Vid.* Salmon's Geometry of Three Dimensions, p. 291.

[†] Conf. note at the bottom of page 157; also Cayley in Quarterly Journal, Vol. 7, and Smith in Proceedings London Math. Society, Vol. 6.

1. *The sufficient and necessary condition that the point* $(x)_0$ *be stationary to the* $\varkappa_1$ *degree is the vanishing of all the determinants of the second order in the matrix:*

$$\begin{Vmatrix} a_{10} & a_{20} & a_{30} & a_{40} \\ a_{11} & a_{21} & a_{31} & a_{41} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{1\varkappa_1} & a_{2\varkappa_1} & a_{3\varkappa_1} & a_{4\varkappa_1} \end{Vmatrix}$$

For, by hypothesis, (x_0) is singular in the 1st class to the $\varkappa_1$ order, that is, $A_0 \equiv A_1, A_2, \ldots A_{\varkappa_1}$.[*] Therefore each of the products $A_0A_1, \ldots A_0A_{\varkappa_1}$, etc., must vanish.

But $A_0A_1 = [a_{10}E_1 + a_{20}E_2 + a_{30}E_3 + a_{40}E_4]\ [a_{11}E_1 + a_{21}E_2 + a_{31}E_3 + a_{41}E_4]$

$$= \begin{vmatrix} a_{10} & a_{20} \\ a_{11} & a_{21} \end{vmatrix} E_1E_2 + \begin{vmatrix} a_{10} & a_{30} \\ a_{11} & a_{31} \end{vmatrix} E_1E_3 + \begin{vmatrix} a_{10} & a_{40} \\ a_{11} & a_{41} \end{vmatrix} E_1E_4$$

$$+ \begin{vmatrix} a_{30} & a_{40} \\ a_{31} & a_{41} \end{vmatrix} E_3E_4 + \begin{vmatrix} a_{40} & a_{20} \\ a_{41} & a_{21} \end{vmatrix} E_4E_2 + \begin{vmatrix} a_{20} & a_{30} \\ a_{21} & a_{31} \end{vmatrix} E_2E_3.$$

The points $E_1E_2E_3E_4$ are linearly independent, so that A_0A_1 can only be 0 when all the determinants $\begin{vmatrix} a_{10} & a_{20} \\ a_{11} & a_{21} \end{vmatrix}$, etc., vanish, and the same reasoning applies to all the remaining products $A_0A_{\varkappa_1}$, etc.

2. *That, in addition to being itself stationary to the* $\varkappa_1$ *degree, its tangent line be stationary to the* $\varkappa_2$ *degree, the necessary and sufficient conditions are* 1, *and the vanishing of all the determinants of the third order in the matrix:*

$$\begin{Vmatrix} a_{10} & a_{20} & a_{30} & a_{40} \\ a_{1,\varkappa_1+1} & a_{2,\varkappa_1+1} & a_{3,\varkappa_1+1} & a_{4,\varkappa_1+1} \\ a_{1,\varkappa_1+2} & a_{2,\varkappa_1+2} & a_{3,\varkappa_1+2} & a_{4,\varkappa_1+2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{1,\varkappa_1+\varkappa_2+1} & a_{2,\varkappa_1+\varkappa_2+1} & a_{3,\varkappa_1+\varkappa_2+1} & a_{4,\varkappa_1+\varkappa_2+1} \end{Vmatrix}$$

For the hypothesis requires that all the products

$$A_0A_{\varkappa_1+1}A_{\varkappa_1+2}, \ldots A_0A_{\varkappa_1+1}A_{\varkappa_1+\varkappa_2+1}$$

vanish, and that can only be when the determinants of the third order in the matrix 2 all vanish.

3. *That, in addition to the singularities* $\varkappa_1, \varkappa_2$, *it may have the further singularity that its osculating plane be stationary to the* $\varkappa_3$ *degree, the sufficient and necessary*

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[*] *Vid.* §1.

conditions are 1, 2, *and the vanishing of all the determinants of the fourth order in the matrix:*

$$\left\|\begin{array}{llll} a_{10} & a_{20} & a_{30} & a_{40} \\ a_{1,\,\kappa_1+1} & a_{2,\,\kappa_1+1} & a_{3,\,\kappa_1+1} & a_{4,\,\kappa_1+1} \\ a_{1,\,\kappa_1+\kappa_2+2} & a_{2,\,\kappa_1+\kappa_2+2} & a_{3,\,\kappa_1+\kappa_2+2} & a_{4,\,\kappa_1+\kappa_2+2} \\ a_{1,\,\kappa_1+\kappa_2+3} & a_{2,\,\kappa_1+\kappa_2+3} & a_{3,\,\kappa_1+\kappa_2+3} & a_{4,\,\kappa_1+\kappa_2+3} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,\,\kappa_1+\kappa_2+\kappa_3+2} & a_{2,\,\kappa_1+\kappa_2+\kappa_3+2} & a_{3,\,\kappa_1+\kappa_2+\kappa_3+2} & a_{4,\,\kappa_1+\kappa_2+\kappa_3+2} \end{array}\right\|$$

1, 2, 3, express equally well the conditions of plane singularity: it is only necessary to suppose the given equations to be

$$u_1 : u_2 : u_3 : u_4 = \psi_1(t) : \psi_2(t) : \psi_3(t) : \psi_4(t).$$

The line singularities admit also of expressions of this sort, but for the general case they are very cumbrous and it does not seem worth while to give them.

§ 8.

If P_t be the Grassmann equation of the point-curve of double curvature

$$x_1 : x_2 : x_3 : x_4 = \phi_1(t) : \phi_2(t) : \phi_3(t) : \phi_4(t),$$

P_t', P_t'', etc., are the equations in the same analysis of the "differential curves"*

$$x_1 : x_2 : x_3 : x_4 = \phi_1'(t) : \phi_2'(t) : \phi_3'(t) : \phi_4'(t)\text{ ; etc.}$$

So also are ε_t', ε_t'', etc., the equations of the first, second, etc., differential (plane-) curves of ε_t.

No addition is made to our knowledge of singularities by regarding P_t', P_t'', etc., in this light, but it leads to a graphical representation of the conditions of singularity which is worth noticing.

Call the curve of which P_t^j is the point equation ${}_{P-}C^j$, the curve of which ε_t^j ($\varepsilon_t = P_t P_t^{\kappa_1+1} P_t^{\kappa_1+\kappa_2+2}$) is the plane equation ${}_{\varepsilon-}C^j$, and understand by "corresponding" elements of the curves such as are determined by the same value of t. Then

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*A familiar example of such a differential curve is Hamilton's hodograph. If in the expansion $P_t = A_0 + A_1(t-t_0) + A_2(t-t_0)^2 +$ etc., only A_0 is a point and the rest of the coefficients, A_1, etc., are vectors (Strecken), the vector P_t' represents, both in direction and magnitude, the velocity of P_t; and if one extremity of it be fixed, the other will trace the hodograph of P_t. Or the curve which P_t' traces in the plane at infinity is the hodograph, the position of the points of this curve representing the direction, their weight the magnitude, of the velocity, of P_t. Conf. Grassmann, Ausdehnungslehre (1884), p. 146.

In a point of which the singularity indices are $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, a curve C is met by ${}_{P-}C'$, ${}_{P-}C''$, ... ${}_{P-}C^{\kappa_1}$ in the corresponding points of these curves; its tangent line passes through the corresponding points of ${}_{P-}C^{\kappa_1+1}$, ... ${}_{P-}C^{\kappa_1+\kappa_2+1}$; and its osculating plane through the corresponding points of ${}_{P-}C^{\kappa_1+\kappa_2+2}$, ... ${}_{P-}C^{\kappa_1+\kappa_2+\kappa_3+2}$: and conversely.

With a plane element of C of which the singularity indices are $\varkappa_3$, $\varkappa_2$, $\varkappa_1$, the corresponding planes of ${}_{\epsilon-}C'$, ... ${}_{\epsilon-}C^{\kappa_3}$ coincide; through its tangent line pass the corresponding planes of ${}_{\epsilon-}C^{\kappa_3+1}$, ... ${}_{\epsilon-}C^{\kappa_3+\kappa_2+1}$; and through its point of osculation the corresponding planes of ${}_{\epsilon-}C^{\kappa_3+\kappa_2+2}$, ... ${}_{\epsilon-}C^{\kappa_3+\kappa_2+\kappa_1+2}$: and conversely.

These corresponding points of ${}_{P-}C'$, ${}_{P-}C''$, etc., have singularities of which the indices are determined within certain limits by $\varkappa_1$, $\varkappa_2$, $\varkappa_3$. For the first $\varkappa_1$ of them the indices are $\varkappa_1 - 1, \varkappa_2, \varkappa_3$; $\varkappa_1 - 2, \varkappa_2, \varkappa_3$; ...; $0, \varkappa_2, \varkappa_3$ respectively. The rest are not determined absolutely; *e. g.* of the point on ${}_{P-}C^{\kappa_1+1}$ it is only required that the sum of its first and second-class indices be $\varkappa_2 - 1$.

The curves ${}_{P-}C'$, ... ${}_{P-}C^{\kappa_1}$ have contacts of the orders $\varkappa_1 + \varkappa_2, \ldots \varkappa_2 + 1$ with the tangent to C in the point $P(\varkappa_1, \varkappa_2, \varkappa_3)$, and contacts of the orders $\varkappa_1 + \varkappa_2 + \varkappa_3, \ldots, \varkappa_2 + \varkappa_3 + 1$ with its osculating plane. The curves ${}_{P-}C^{\kappa_1+1}$, ... ${}_{P-}C^{\kappa_1+\kappa_2}$ also are generally touched by this tangent and the curves ${}_{P-}C^{\kappa_1+1}$ ... ${}_{P-}C^{\kappa_1+\kappa_2+\kappa_3}$ by this osculating plane. If, however, the point corresponding to P on one of them, ${}_{P-}C^{j}$, have the first-class singularity index $\varkappa_2 + \varkappa_1 + 1 - j$, it and the curves ${}_{P-}C^{j+1}$, ... ${}_{P-}C^{\kappa_1+\kappa_2+1}$ are touched by the line joining those points of ${}_{P-}C^{j}$ and ${}_{P-}C^{\kappa_1+\kappa_2+2}$ which correspond to P. Even in this case the curves ${}^{P-}C'$, ... ${}_{P-}C^{\kappa_1+\kappa_2}$ are touched by the osculating plane of C, unless the corresponding point on ${}_{P-}C^{\kappa_1+\kappa_2+2}$ have the first-class singularity index $\varkappa_3$. It is evident that the curves ${}_{P-}C^{\kappa_1+\kappa_2+j}$ ... ${}_{P-}C^{\kappa_1+\kappa_2+\kappa_3}$ also are not touched by the osculating plane of C if the first-class singularity index of the point on ${}_{P-}C^{\kappa_1+\kappa_2+j}$ be $\varkappa_3 - j + 1$, or the sum of its first and second-class singularity indices $\varkappa_3 - j + 2$.

The projections of ${}_{P-}C'$ ... ${}_{P-}C^{\kappa_1}$ upon the osculating plane of C at P touch the projection of C upon the same plane; the first, third, fifth, etc., of them crossing the projection of C, the second, fourth, sixth, etc., not crossing it.

There are reciprocal theorems for the plane-curves ${}_{\epsilon-}C'$, ${}_{\epsilon-}C''$, etc.

${}_{P-}C'$, ${}_{P-}C''$, etc., and ${}_{\epsilon-}C'$, ${}_{\epsilon-}C''$, etc., are two distinct sets of curves. By constructing the point equations of ${}_{\epsilon-}C'$, ${}_{\epsilon-}C''$, etc., it is found that all of these curves to ${}_{\epsilon-}C^{\kappa_1+\kappa_2+\kappa_3}$ inclusive meet C in $P(\varkappa_1, \varkappa_2, \varkappa_3)$; but that the corresponding point of the envelope of ${}_{\epsilon-}C^{\kappa_1+\kappa_2+\kappa_3+1}$ may lie anywhere in space.

LEIPSIC, May 27, 1885.